

The flat limit of three dimensional asymptotically anti-de Sitter spacetimes

Glenn Barnich*

*Physique Théorique et Mathématique, Université Libre de Bruxelles and International Solvay Institutes,
Campus Plaine C.P. 231, B-1050 Bruxelles, Belgium*

Andrés Gomberoff†

*Universidad Andres Bello, Departamento de Ciencias Físicas,
Av. República 252, Santiago, Chile.*

Hernán A. González‡

Departamento de Física, P. Universidad Católica de Chile, Casilla 306, Santiago 22, Chile

In order to get a better understanding of holographic properties of gravitational theories with a vanishing cosmological constant, we analyze in detail the relation between asymptotically anti-de Sitter and asymptotically flat spacetimes in three dimensions. This relation is somewhat subtle because the limit of vanishing cosmological constant cannot be naively taken in standard Fefferman-Graham coordinates. After reformulating the standard anti-de Sitter results in Robinson-Trautman coordinates, a suitably modified Penrose limit is shown to connect both asymptotic regimes.

* gbarnich@ulb.ac.be; Research Director of the Fund for Scientific Research Belgium

† agomberoff@unab.cl

‡ hdgonzal@uc.cl

I. INTRODUCTION

Even though quantum properties are still not completely understood [1, 2], on the classical and semi-classical level asymptotically anti-de Sitter spacetimes in three dimensions constitute an extremely rich and well-studied framework:

1. As an early precursor to the AdS/CFT correspondence [3, 4], their symmetry algebra has been shown to consist of two commuting copies of centrally non-extended Virasoro algebras with a central extension arising in the Dirac bracket algebra of the canonical generators [5]. The value of the central charges $c^\pm = \frac{3l}{2G}$ has been used to argue for a microscopic derivation of the Bekenstein-Hawking entropy of the black holes [6], independently of the details of the underlying theory [7, 8].
2. The general solution to the equations of motion is known in closed form [9, 10] in Fefferman-Graham coordinates, it includes spinning black holes [11], and all other solutions can be obtained from this 2 parameter family through suitable coordinate transformations [12].
3. Additional control on holographic properties comes from the Chern-Simons formulation of three dimensional anti-de Sitter gravity [13, 14] and the relation of Chern-Simons theories with conformal field theories on their boundary [15–17]. In particular, the asymptotic dynamics has been understood from this point of view in [18, 19] (see also [20]).

In view of these results, a valid strategy to get additional insight into holographic properties of gravitational theories with vanishing cosmological constant [21–25] is to study in more detail asymptotically flat spacetimes in three dimensions and their relation to the asymptotically anti-de Sitter case.

The symmetry algebra \mathfrak{bms}_3 of asymptotically flat three dimensional spacetimes [26] involves both a supertranslation and a superrotation sub-algebra, the latter being given by a centrally non-extended Virasoro algebra. The Dirac bracket algebra of the surface charges has a central charge with value $c = \frac{3}{G}$ between the superrotation and supertranslation generators [27], which is related to the anti-de Sitter algebra through a suitable redefinition of the generators followed by taking the cosmological constant to zero. Furthermore, using the three dimensional analog of the four-dimensional Bondi-Metzner-Sachs (BMS) gauge [28, 29], the general solution to the equations of motion can also be found in closed form [30]. It involves two arbitrary functions of one variable, exactly as in the anti-de Sitter case. Finally, from the Chern-Simons point of view, some aspects of the boundary dynamics of flat space gravity have been discussed in [31].

In this paper we study in detail the suitably modified Penrose limit that connects the general solution, symmetries and surface charges of asymptotically AdS_3 and Minkowski spacetimes in three dimensions.

The plan of the paper is as follows. In the next section, we briefly recall standard results on asymptotic symmetries, solutions and charges in the anti-de Sitter case in Feffermann-Graham coordinates and show that the limit $l \rightarrow \infty$ cannot be performed in a straightforward way. In section III, we re-derive the anti-de Sitter results in the BMS gauge. The procedure for taking the limit is explained in detail in section IV. In section V, we briefly discuss the simplest geometries when only the zero modes are excited.

The flat limit from the point of view of the Chern-Simons formulation, including a non-relativistic \mathfrak{bms}_3 invariant Liouville theory, will be discussed elsewhere. More generally, the representation theory for \mathfrak{bms}_3 needs to be studied in more detail. An investigation using the fact that \mathfrak{bms}_3 is isomorphic to \mathfrak{gca}_2 , a non-relativistic contraction of the Virasoro algebra, can be found in [32–34].

II. SYMMETRIES, SOLUTIONS AND CHARGES IN THE FEFFERMAN-GRAHAM GAUGE

An asymptotically AdS_3 metric in the spirit of Fefferman-Graham consists in a metric ansatz

$$ds^2 = \frac{l^2}{\rho^2} d\rho^2 + g_{AB}(\rho, x) dx^A dx^B, \quad (1)$$

where the negative cosmological constant is $\Lambda = -1/l^2$. The Einstein equations of motion then imply in particular that

$$g_{AB} = \rho^2 \bar{\gamma}_{AB} + O(1), \quad (2)$$

and we take for simplicity $\bar{\gamma}_{AB} = \eta_{AB} = \text{diag}(-1, 1)$, the flat metric on the cylinder $x^A = (\frac{t}{l}, \phi)$ in what follows. It will also be useful to use light-cone coordinates $x^\pm = \frac{t}{l} \pm \phi$.

The infinitesimal transformations leaving the form (1) and (2) of the metric invariant are generated by

$$\begin{cases} \xi^\rho = -\frac{1}{2}\psi\rho, \\ \xi^A = Y^A + I^A, \quad I^A = -\frac{l^2}{2}\partial_B\psi\int_\rho^\infty \frac{d\rho'}{\rho'}g^{AB}, \end{cases} \quad (3)$$

where Y^A is a conformal Killing vector of $\bar{\gamma}_{AB}dx^A dx^B = -dx^+ dx^-$, while $\psi = \bar{D}_A Y^A$ is the conformal factor. These spacetime vectors form a representation of the algebra of conformal Killing vectors of $\bar{\gamma}_{AB}dx^A dx^B$,

$$[\xi_{Y_1}, \xi_{Y_2}]_M^\mu \equiv [\xi_{Y_1}, \xi_{Y_2}]^\mu - \delta_{\xi_{Y_1}}^g \xi_{Y_2}^\mu + \delta_{\xi_{Y_2}}^g \xi_{Y_1}^\mu = \xi_{[Y_1, Y_2]}^\mu. \quad (4)$$

On the cylinder, one can expand $Y^\pm(x^\pm) = \sum_{m \in \mathbb{Z}} Y_m^\pm e^{-imx^\pm}$, with $\bar{Y}_m^\pm = Y_{-m}^\pm$. For the generators $l_m^+ = \xi_{e^{imx^+}, 0}^\mu \partial_\mu$, $l_m^- = \xi_{0, e^{imx^-}}^\mu \partial_\mu$ one gets

$$i[l_m^\pm, l_n^\pm]_M = (m-n)l_{m+n}^\pm, \quad [l_m^\pm, l_n^\mp]_M = 0. \quad (5)$$

When requiring that $\bar{\gamma}_{AB} = \eta_{AB}$, the general solution to Einstein's equations is given by

$$ds^2 = \frac{l^2}{\rho^2} d\rho^2 - (\rho dx^+ - \frac{l^2}{\rho} \Xi_{--} dx^-)(\rho dx^- - \frac{l^2}{\rho} \Xi_{++} dx^+), \quad (6)$$

where $\Xi_{\pm\pm} = \Xi_{\pm\pm}(x^\pm)$. The easiest solutions where these functions are constants

$$\Xi_{\pm\pm}^{BTZ} = 2G(M \pm \frac{J}{l}), \quad (7)$$

include both the BTZ black holes for which $M \geq 0$, $|J| \leq Ml$ and AdS_3 which corresponds to $M = -\frac{1}{8G}$, $J = 0$.

The action of the transformations generated by ξ_Y on solution space reads

$$-\delta_Y \Xi_{\pm\pm} = Y^\pm \partial_\pm \Xi_{\pm\pm} + 2\partial_\pm Y^\pm \Xi_{\pm\pm} - \frac{1}{2} \partial_\pm^3 Y^\pm. \quad (8)$$

The conserved surface charges, computed with respect to the AdS_3 background¹, are given by

$$Q_Y = \frac{l}{8\pi G} \int_0^{2\pi} d\phi \left[Y^+ (\Xi_{++} + \frac{1}{4}) + Y^- (\Xi_{--} + \frac{1}{4}) \right]. \quad (9)$$

The generators are denoted by $L_m^+ = Q_{e^{imx^+}, 0}$, $L_m^- = Q_{0, e^{imx^-}}$ and for the BTZ black hole we find in particular

$$\frac{1}{l} (L_m^{+BTZ} + L_m^{-BTZ}) = \delta_m^0 (M + \frac{1}{8G}), \quad L_m^{+BTZ} - L_m^{-BTZ} = \delta_0^m J, \quad (10)$$

the non-vanishing value being associated with $\partial_\tau = l\partial_t$ and with ∂_ϕ respectively. More generally, up to normalization and a shift of the zero mode, the charges L_{-m}^\pm are the coefficients of the Fourier expansion of $\Xi_{\pm\pm}$,

$$\Xi_{\pm\pm} = -\frac{1}{4} + \sum_m \frac{4G}{l} L_m^\pm e^{-imx^\pm}. \quad (11)$$

According to general results from [5, 35–37] the Dirac bracket of the surface charges is taken to be

$$\{Q_{Y_1}, Q_{Y_2}\} = \delta_{Y_1} Q_{Y_2}. \quad (12)$$

For the generators, one then gets

$$i\{L_m^\pm, L_n^\pm\} = (m-n)L_{m+n}^\pm + \frac{c^\pm}{12} m(m^2-1)\delta_{m+n}^0, \quad \{L_m^\pm, L_n^\mp\} = 0, \quad (13)$$

where

$$c^\pm = \frac{3l}{2G}. \quad (14)$$

¹ In [30] an overall factor of l was missed, while the charges were computed with respect to the $M = 0 = J$ BTZ black hole.

After quantization (with $\hbar = 1$), the commutator of the associated quantum operators in the limit where $c^\pm \gg 1$ is given by

$$[L_m^\pm, L_n^\pm] = (m-n)L_{m+n}^\pm + \frac{c^\pm}{12}m(m^2-1)\delta_{m+n}^0, \quad [L_m^\pm, L_n^\mp] = 0. \quad (15)$$

By defining the generators

$$P_m = \frac{1}{l}(L_m^+ + L_{-m}^-), \quad J_m = L_m^+ - L_{-m}^-, \quad (16)$$

the surface charge algebra becomes

$$\begin{aligned} i\{J_m, J_n\} &= (m-n)J_{m+n} + \frac{c^+ - c^-}{12}m(m^2-1)\delta_{m+n}^0, \\ i\{J_m, P_n\} &= (m-n)P_{m+n} + \frac{c^+ + c^-}{12\ell}m(m^2-1)\delta_{m+n}^0, \\ i\{P_m, P_n\} &= \frac{1}{\ell^2}((m-n)J_{m+n} + \frac{c^+ - c^-}{12}m(m^2-1)\delta_{m+n}^0). \end{aligned} \quad (17)$$

In the purely gravitational case with central charges as in (14), the limit $l \rightarrow \infty$ is well-defined and gives rise to the centrally extended \mathfrak{bms}_3 algebra [38],

$$\begin{aligned} i\{J_m, J_n\} &= (m-n)J_{m+n} + \frac{c_1}{12}m(m^2-1)\delta_{m+n}^0, \\ i\{J_m, P_n\} &= (m-n)P_{m+n} + \frac{c_2}{12}m(m^2-1)\delta_{m+n}^0, \\ i\{P_m, P_n\} &= 0, \end{aligned} \quad (18)$$

where

$$c_1 = 0, \quad c_2 = \frac{3}{G}. \quad (19)$$

This agrees with the results found by a direct computation for asymptotically flat Einstein gravity in three dimensions in [27].

Note, however, that the line element (6) is not well defined in the limit $l \rightarrow \infty$. This is not entirely surprising since taking limits in spacetime geometries is quite a subtle issue [39]. In particular, results depend on the coordinates that are held fixed during the limit.

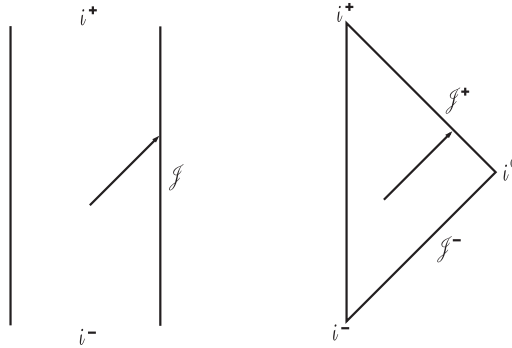


FIG. 1. Penrose diagrams of anti-de Sitter and Minkowski spacetimes. Arrows in the diagrams represent outgoing null rays.

III. RESULTS IN THE BMS GAUGE

In order to be able to relate the asymptotic analysis in both cases we have to take into account that in flat space, the analysis is performed at null infinity. For definiteness, we concentrate on future null infinity \mathcal{J}^+ . Since gauge

fixing is independent of the presence of a cosmological constant, we can choose the three dimensional analog of the BMS gauge for both asymptotics. This can be done by making the following metric ansatz in terms of coordinates u, r, ϕ ,

$$ds^2 = e^{2\beta} \frac{V}{r} du^2 - 2e^{2\beta} du dr + r^2 (d\phi - U du)^2, \quad (20)$$

for three arbitrary functions β, V, U . For instance, defining the retarded time u through $t = u + l \arctan \frac{r}{l}$, the AdS_3 metric in global coordinates,

$$ds^2 = -\left(1 + \frac{r^2}{l^2}\right) dt^2 + \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\phi^2, \quad (21)$$

is of the form (20) with $\beta = 0 = U$, $\frac{V}{r} = -\frac{r^2}{l^2} - 1$. Furthermore, the limit $l \rightarrow \infty$ can be safely taken and yields Minkowski spacetime, as desired.

Assuming that $\beta = o(1) = U$, the Einstein equations of motion imply in particular that

$$\frac{V}{r} = -\frac{r^2}{l^2} + O(1), \quad \beta = O(r^{-1}), \quad U = O(r^{-2}). \quad (22)$$

Infinitesimal transformations that keep the gauge fixed form (20) invariant are generated by vector fields ξ^μ such that

$$\mathcal{L}_\xi g_{rr} = 0 = \mathcal{L}_\xi g_{r\phi} = \mathcal{L}_\xi g_{\phi\phi}, \quad (23)$$

and are explicitly given by

$$\xi^u = f, \quad \xi^\phi = Y - \partial_\phi f \int_r^\infty dr' r'^{-2} e^{2\beta}, \quad \xi^r = -r(\partial_\phi \xi^\phi - U \partial_\phi f), \quad (24)$$

where $\partial_r f = \partial_r Y = 0$. When requiring in addition that the fall-off conditions (22) be preserved,

$$\mathcal{L}_\xi g_{ur} = O(r^{-1}), \quad \mathcal{L}_\xi g_{u\phi} = O(1), \quad \mathcal{L}_\xi g_{uu} = O(1). \quad (25)$$

the additional conditions are

$$\partial_u f = \partial_\phi Y, \quad \partial_u Y = \frac{1}{l^2} \partial_\phi f, \quad (26)$$

and hence

$$f = \frac{l}{2}(Y^+ + Y^-), \quad Y = \frac{1}{2}(Y^+ - Y^-), \quad (27)$$

where now $x^\pm = \frac{u}{l} \pm \phi$, and $Y^\pm = Y^\pm(x^\pm)$ are arbitrary functions of their arguments. It now follows that (4), or equivalently (5), also hold in the BMS gauge: the spacetime vectors (24) form a representation of the conformal Lie algebra on an asymptotically AdS_3 spacetime defined through (20) and (22) when equipped with the modified bracket.

In the gauge (20) assuming furthermore that $\beta = o(1) = U$, the most general solution to the Einstein equations $G_{\alpha\beta} = l^{-2} g_{\alpha\beta}$ is easily worked out. One finds

$$\beta = 0, \quad U = -r^{-2} \mathcal{N}, \quad \frac{V}{r} = -\frac{r^2}{l^2} + \mathcal{M} - r^{-2} \mathcal{N}^2, \quad (28)$$

or equivalently

$$ds^2 = \left(-\frac{r^2}{l^2} + \mathcal{M}\right) du^2 - 2du dr + 2\mathcal{N} du d\phi + r^2 d\phi^2, \quad (29)$$

where $\partial_r \mathcal{M} = 0 = \partial_r \mathcal{N}$. In addition,

$$\partial_u \mathcal{M} = \frac{2}{l^2} \partial_\phi \mathcal{N}, \quad 2\partial_u \mathcal{N} = \partial_\phi \mathcal{M}, \quad (30)$$

so that

$$\mathcal{M}(u, \phi) = 2(\Xi_{++} + \Xi_{--}), \quad \mathcal{N}(u, \phi) = l(\Xi_{++} - \Xi_{--}), \quad (31)$$

where $\Xi_{\pm\pm} = \Xi_{\pm\pm}(x^\pm)$. Note that on-shell, the asymptotic Killing vectors become

$$\xi^u = f, \quad \xi^\phi = Y - r^{-1}\partial_\phi f, \quad \xi^r = -r\partial_\phi Y + \partial_\phi^2 f - r^{-1}\mathcal{N}\partial_\phi f. \quad (32)$$

In these coordinates, the BTZ black hole is again determined by (7), while computing $\mathcal{L}_\xi g_{\mu\nu}$, with ξ as in (32) and $g_{\mu\nu}$ as in (29) and (31), gives the transformation laws (8).

The charges are computed along the lines of [37] at the circle at infinity u constant and $r \rightarrow \infty$ using as background the AdS_3 space-time in the form

$$d\bar{s}^2 = -\left(1 + \frac{r^2}{l^2}\right)du^2 - 2dudr + r^2d\phi^2. \quad (33)$$

In a first stage, this gives

$$Q_\xi[h, \bar{g}] = \frac{1}{16\pi G} \int_0^{2\pi} d\phi [f(\mathcal{M} + 1) + 2Y\mathcal{N}], \quad (34)$$

and then, using (27) and (31), one recovers (9). It follows that all relations of the previous section discussed after equation (9) hold in the BMS gauge as well.

IV. FLAT LIMIT

We now study the flat limit of asymptotically AdS_3 space-times in the BMS gauge. More precisely, we want to take the limit of the cosmological constant going to zero in such a way that asymptotically AdS_3 solutions get mapped to asymptotically flat solutions. This will be achieved through a suitably modified Penrose limit.

The first observation is that asymptotically AdS_3 spacetimes in the BMS gauge belong to the Robinson-Trautman class of metrics [40], for which u parametrizes null hypersurfaces and r is an affine parameter for the associated null geodesic generators. It is in this setting that Penrose has shown that a particular limit of a generic spacetime is given by a plane wave [41]. Second, in the context of string theory backgrounds, Penrose limits were used to connect backgrounds with AdS factors to backgrounds involving flat or plane wave backgrounds [42, 43].

Consider the Einstein-Hilbert action for three dimensional anti-de Sitter gravity,

$$S[g; G, l] = \frac{1}{16\pi G} \int d^3x \sqrt{|g|} \left(R + \frac{2}{l^2}\right). \quad (35)$$

For any metric g and $\lambda > 0$, let $g^{(\lambda)} = \lambda^{-2}g$ denote the suitably rescaled metric. If $G^{(\lambda)} = \lambda^{-1}G$, $l^{(\lambda)} = \lambda^{-1}l$, we have

$$S[g^{(\lambda)}; G^{(\lambda)}, l^{(\lambda)}] = S[g; G, l]. \quad (36)$$

Consider then a family of spacetimes $g(\lambda)$ labelled by a parameter $\lambda > 0$ that is a solution to the Einstein equations with cosmological constant $\Lambda = -\frac{1}{l^2}$, $R_{\mu\nu}[g(\lambda)] = -2l^{-2}g_{\mu\nu}(\lambda)$. In particular, according to the previous section, the most general asymptotically AdS_3 family of solutions of this type is obtained by using $\Xi_{\pm\pm}(\lambda) = \Xi_{\pm\pm}(x^\pm, \lambda)$ with an arbitrary λ dependence instead of $\Xi_{\pm\pm}(x^\pm)$ in (31) and (29). It then follows from (36) that $g^{(\lambda)}(\lambda)$ is a solution to the Einstein equations with cosmological radius $l^{(\lambda)}$. This can also be seen directly by using

$$R_{\mu\nu}[\lambda^{-2}g(\lambda)] = R_{\mu\nu}[g(\lambda)] = -2\frac{\lambda^2}{l^2}(\lambda^{-2}g_{\mu\nu}(\lambda)). \quad (37)$$

If $g = \lim_{\lambda \rightarrow 0} \lambda^{-2}g(\lambda)$ is a well-defined metric, it thus defines a solution to the Einstein equations with vanishing cosmological constant.

For instance, the Penrose limit [41] of the metrics (29) consists in rescaling the metric as above and simultaneously scaling the coordinates as

$$(u, r, \phi) \rightarrow (\lambda^2 u, r, \lambda \phi). \quad (38)$$

The transformed metric is

$$\lambda^{-2}ds_\lambda^2 = -\lambda^2\left[\frac{r^2}{l^2} - \mathcal{M}(\lambda^2u, \lambda\phi)\right]du^2 - 2dudr + 2\lambda\mathcal{N}(\lambda^2u, \lambda\phi)dud\phi + r^2d\phi^2. \quad (39)$$

If $\mathcal{M}(u, \phi)$ and $\mathcal{N}(u, \phi)$ are continuous functions of the coordinates, the Ricci-flat limiting metric is simply the null orbifold [44]

$$ds^2 = -2dudr + r^2d\phi^2. \quad (40)$$

We would like to perform a different flat limit² which keeps the number of arbitrary functions appearing in the metric. This can be done through the coordinate scaling,

$$(u, r, \phi) \rightarrow (\lambda u, \lambda r, \phi). \quad (41)$$

In this case, the transformed metric is

$$\lambda^{-2}ds_\lambda^2 = \left[-\frac{\lambda^2r^2}{l^2} + \mathcal{M}(\lambda u, \phi)\right]du^2 - 2dudr + 2\lambda^{-1}\mathcal{N}(\lambda u, \phi)dud\phi + r^2d\phi^2. \quad (42)$$

In order to control what happens in the limit, we have to take into account the expression (31) of the functions \mathcal{M}, \mathcal{N} in terms of the arbitrary functions $\Xi_{\pm\pm}(\lambda)$. In terms of modes, this gives

$$\begin{aligned} \mathcal{M}(\lambda u, \phi) &= -1 + 8G \sum_m \left(\frac{L_m^+(\lambda)e^{-im\frac{\lambda u}{l}} + L_{-m}^-(\lambda)e^{im\frac{\lambda u}{l}}}{l} \right) e^{-im\phi}, \\ \lambda^{-1}\mathcal{N}(\lambda u, \phi) &= \frac{4G}{\lambda} \sum_m \left(L_m^+(\lambda)e^{-im\frac{\lambda u}{l}} - L_{-m}^-(\lambda)e^{im\frac{\lambda u}{l}} \right) e^{-im\phi}. \end{aligned} \quad (43)$$

In other words, up to the arbitrary λ dependence in $\Xi_{\pm\pm}$, or equivalently in their Fourier modes, in the coordinates u, r, ϕ , the metric $\lambda^{-2}ds_\lambda^2$ is obtained from (29) simply by replacing $l \rightarrow l^{(\lambda)}$ and $G \rightarrow G^{(\lambda)}$.

In order to have a well-defined limit, we then need that

$$\begin{aligned} L_m^+(\lambda) &= \frac{1}{2}lP_m + \lambda L_m'^+(0) + O(\lambda^2), \\ L_m^-(\lambda) &= \frac{1}{2}lP_{-m} + \lambda L_m'^-(0) + O(\lambda^2). \end{aligned} \quad (44)$$

In this case, the limit becomes

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \mathcal{M}(\lambda u, \phi) &= -1 + 8G \sum_m P_m e^{-im\phi} = \Theta(\phi), \\ \lim_{\lambda \rightarrow 0} \lambda^{-1}\mathcal{N}(\lambda u, \phi) &= 4G \sum_m (J_m - uimP_m) e^{-im\phi} = \Xi(\phi) + \frac{u}{2}\partial_\phi\Theta(\phi), \end{aligned} \quad (45)$$

with

$$P_m = \frac{L_m^+(0) + L_{-m}^-(0)}{l}, \quad J_m = L_m'^+(0) - L_{-m}'^-(0). \quad (46)$$

In summary

$$\lim_{\lambda \rightarrow 0} \lambda^{-2}ds_\lambda^2 = \Theta(\phi)du^2 - 2dudr + 2\left[\Xi(\phi) + \frac{u}{2}\partial_\phi\Theta(\phi)\right]dud\phi + r^2d\phi^2. \quad (47)$$

We have thus recovered through this limit the most general solution to the asymptotically flat Einstein equations as discussed in [30], i.e, Ricci flat metrics of the form (20) with the fall-off conditions as in (22) with $l \rightarrow \infty$ so that $\frac{V}{r} = O(1)$.

The limit can be used, as well, to relate the symmetries of asymptotically AdS_3 and flat spacetimes. The first step is again that for a general asymptotically AdS_3 metric $g(\lambda)$, the most general asymptotic Killing vectors involve

² It should be interesting to analyze such modified Penrose limits in higher dimensions.

the arbitrary functions $Y^\pm(x^\pm, \lambda) = \sum_m Y_{-m}^\pm(\lambda) e^{imx^\pm}$ with an arbitrary dependence on λ in (3) instead of $Y^\pm(x^\pm)$. Under the change of coordinates (41), the asymptotic Killing vectors (32) then acquire the form

$$\xi(\lambda) = \lambda^{-1} \xi^u(\lambda u, \lambda r, \phi) \partial_u + \lambda^{-1} \xi^r(\lambda u, \lambda r, \phi) \partial_r + \xi^\phi(\lambda u, \lambda r, \phi) \partial_\phi, \quad (48)$$

Consider first the leading order parts of $\xi^u(\lambda)$ and $\xi^\phi(\lambda)$,

$$\begin{aligned} \lambda^{-1} f(\lambda u, \phi) &= \frac{l}{2\lambda} \sum_m \left(Y_m^+(\lambda) e^{-im \frac{\lambda u}{l}} + Y_{-m}^-(\lambda) e^{im \frac{\lambda u}{l}} \right) e^{-im\phi}, \\ Y(\lambda u, \phi) &= \frac{1}{2} \sum_m \left(Y_m^+(\lambda) e^{-im \frac{\lambda u}{l}} - Y_{-m}^-(\lambda) e^{im \frac{\lambda u}{l}} \right) e^{-im\phi}. \end{aligned} \quad (49)$$

When taking into account the scaling of r and the previously discussed behaviour of \mathcal{N} in (43), it follows again that, apart from the arbitrary λ dependence in the Fourier modes, in the coordinates u, r, ϕ , $\xi(\lambda)$ is obtained from (32) through the substitution $l \rightarrow l^{(\lambda)}$, $G \rightarrow G^{(\lambda)}$.

A necessary condition for a well defined limit $\lambda \rightarrow 0$ is then

$$\begin{aligned} Y_m^+(\lambda) &= Y_m + \lambda Y_m'^+(0) + O(\lambda^2), \\ Y_m^-(\lambda) &= -Y_{-m} + \lambda Y_m'^-(0) + O(\lambda^2), \end{aligned} \quad (50)$$

so that

$$\begin{aligned} \lim_{\lambda \rightarrow 0} \lambda^{-1} f(\lambda u, \phi) &= \sum_m (T_m - uim Y_m) e^{-im\phi} = T(\phi) + u \partial_\phi Y(\phi), \\ \lim_{\lambda \rightarrow 0} Y(\lambda u, \phi) &= \sum_m Y_m e^{-im\phi} = Y(\phi), \end{aligned} \quad (51)$$

with

$$T_m = \frac{l}{2} (Y_m'^+(0) + Y_{-m}'^-(0)), \quad Y_m = \frac{1}{2} (Y_m^+(0) - Y_{-m}^-(0)). \quad (52)$$

Again, the scaling of r and the previously discussed limit of \mathcal{N} in (45), then implies that $\lim_{\lambda \rightarrow 0} \xi(\lambda) = \xi_{T,Y}^\mu \partial_\mu$ where the components $\xi_{T,Y}^\mu$ are given by (32) with $f = T + u \partial_\phi Y$ and $\mathcal{N} = \Xi + \frac{u}{2} \partial_\phi \Theta$. This limit coincides with the direct computation in [30] of the vectors describing the symmetries of asymptotically flat spacetimes in three dimension.

Let us now turn to the charges. In the presence of λ , the charges $Q_{Y^+, Y^-}(\lambda)$ are given by (34) with $f(u, \phi) \rightarrow \lambda^{-1} f(\lambda u, \phi)$, $Y(u, \phi) \rightarrow Y(\lambda u, \phi)$, $\mathcal{M}(u, \phi) \rightarrow \mathcal{M}(\lambda u, \phi)$ and $\mathcal{N}(u, \phi) \rightarrow \lambda^{-1} \mathcal{N}(\lambda u, \phi)$. In the limit, one finds

$$\lim_{\lambda \rightarrow 0} Q_{Y^+, Y^-}(\lambda) = \frac{1}{16\pi G} \int_0^{2\pi} d\phi [T(\Theta + 1) + 2Y\Xi] = Q_{T,Y} \quad (53)$$

in agreement with the direct computation in [30]³. In terms of modes, we have

$$Q_{Y^+, Y^-}(\lambda) = \lambda^{-1} \sum_m (Y_{-m}^+(\lambda) L_m^+(\lambda) + Y_{-m}^-(\lambda) L_m^-(\lambda)) \xrightarrow{\lambda \rightarrow 0} \sum_m (T_{-m} P_m + Y_{-m} J_m), \quad (54)$$

by taking into account (46) and (52).

In the limit, the algebra of the vectors $\xi_{T,Y}^\mu \partial_\mu$ in the Lie algebroid bracket $[\cdot, \cdot]_M$ has been computed directly and shown to form a representation of \mathfrak{bms}_3 . More concretely, in terms of this bracket, the vectors $t_m = \xi_{e^{im\phi}, 0}^\mu \partial_\mu$, $l_m = \xi_{0, e^{im\phi}}^\mu \partial_\mu$ satisfy (18) with $c_1 = 0 = c_2$. Similarly, the Dirac bracket algebra has also been computed directly in the limit. If $P_m = Q_{e^{im\phi}, 0}$ and $J_m = Q_{0, e^{im\phi}}$, their Dirac bracket algebra is (18) with central charges given in (19). It thus follows that the asymptotic symmetry algebra and its representation get contracted in the limiting procedure, as also observed for exact isometries in [43] for instance.

For completeness, let us finish by showing how the contraction of the algebra occurs concretely in this case. For definiteness, we will concentrate on the vector fields

$$v_{Y^+, Y^-} = f \partial_u + Y \partial_\phi, \quad (55)$$

³ Note that in the last line of (3.18) of this reference Θ should be replaced by $\Theta + 1$.

with f, Y given in terms of Y^+, Y^- as in (27). These vectors fields form a representation of (two commuting copies) of the conformal algebra in the standard Lie bracket. As before, besides the arbitrary λ dependence, in the coordinates u, ϕ , $v_{Y^+, Y^-}(\lambda)$ is obtained by replacing $l \rightarrow l^{(\lambda)}$. It then easily follows that $v_{Y^+, Y^-}(\lambda)$ also form a representation of the conformal algebra,

$$[v_{Y_1^+, Y_1^-}(\lambda), v_{Y_2^+, Y_2^-}(\lambda)] = v_{[Y_1, Y_2]^+, [Y_1, Y_2]^-}(\lambda). \quad (56)$$

This can be seen in the coordinates $x^\pm(\lambda) = \frac{u\lambda}{l} \pm \phi$, where $v_{Y^+, Y^-}(\lambda) = Y^+(\lambda, x^+(\lambda))\partial_{x^+(\lambda)} + Y^-(\lambda, x^+(\lambda))\partial_{x^-(\lambda)}$. In particular, on the right hand side of (56), we have $[Y_1, Y_2]^\pm(\lambda) = Y_1^\pm(\lambda, x^\pm(\lambda))\partial_{x^\pm(\lambda)}Y_2^\pm((\lambda, x^\pm(\lambda)) - (1 \leftrightarrow 2)$. We have already shown that, when the expansion (50) holds, $\lim_{\lambda \rightarrow 0} v_{Y_1, Y_2}(\lambda) = v_{T, Y}$ with $v_{T, Y}$ of the form (55) with $f = T + u\partial_\phi Y$ and T, Y arbitrary functions of ϕ . We then need to evaluate the right hand side. Before taking the limit, the u and ϕ components are given by

$$\begin{aligned} & \frac{l}{2\lambda} \sum_{n,m} i(m-n) \left[Y_{m1}^+(\lambda) Y_{n2}^+(\lambda) e^{-i(m+n)\frac{u\lambda}{l}} - Y_{-m1}^-(\lambda) Y_{-n2}^-(\lambda) e^{i(m+n)\frac{u\lambda}{l}} \right] e^{-i(m+n)\phi}, \\ & \frac{1}{2} \sum_{n,m} i(m-n) \left[Y_{m1}^+(\lambda) Y_{n2}^+(\lambda) e^{-i(m+n)\frac{u\lambda}{l}} + Y_{-m1}^-(\lambda) Y_{-n2}^-(\lambda) e^{i(m+n)\frac{u\lambda}{l}} \right] e^{-i(m+n)\phi}. \end{aligned} \quad (57)$$

Using (50), taking the limit and reconstructing the functions gives $Y_1\partial_\phi T_2 + T_1\partial_\phi Y_2 + u\partial_\phi(Y_1\partial_\phi Y_2) - (1 \leftrightarrow 2)$ and $Y_1\partial_\phi Y_2 - (1 \leftrightarrow 2)$, as one should for the \mathfrak{bms}_3 algebra.

V. ZERO MODES

Let us now concentrate on the zero modes. In the AdS case, from (29), (31), in the parametrization in terms of M and J as in (7), we thus have $\mathcal{M} = 8GM$, $\mathcal{N} = 4GJ$ with charges

$$Q_{\partial_u} = M + \frac{1}{8G}, \quad Q_{\partial_\phi} = J. \quad (58)$$

This metric is explicitly related to the standard ADM form,

$$\begin{aligned} ds^2 &= -N^2 dt^2 + N^{-2} dr^2 + r^2 (d\varphi + N^\varphi dt)^2, \\ N^2 &= \frac{r^2}{l^2} - 8MG + \frac{16G^2 J^2}{r^2}, \quad N^\varphi = \frac{4GJ}{r^2}, \end{aligned} \quad (59)$$

through $t = u + f(r)$, $\varphi = \phi + g(r)$, where $f' = N^{-2}$, $g' = -N^\varphi f'$. In turn, the Fefferman-Graham form (6) for constant $\Xi_{\pm\pm}$ is then obtained through $x^\pm = \frac{t}{l} \pm \varphi$ and $\frac{r^2}{l^2} = \frac{\rho^2}{l^2} + (\Xi_{++} + \Xi_{--}) + \frac{l^2}{\rho^2} \Xi_{++} \Xi_{--}$.

Let us first comment briefly on the space of solutions with non-vanishing negative cosmological constant, depicted in Fig. 2a. The BTZ black holes correspond to the region $M \geq \frac{|J|}{l}$. Geometries satisfying $0 < M < \frac{|J|}{l}$ leave exposed the chronological singularity at $r = 0$ which encloses an unbounded region, $r < 0$, containing closed time-like curves [45]. On the other hand, for $-\frac{1}{8G} < M \leq -\frac{|J|}{l}$, the geometry describes a spinning particle sitting at $r = 0$ [46], which produces a conical defect around it (see also [47]). In the neighborhood of the particle, there is, in general, a bounded region containing closed timelike curves. When $M = -\frac{1}{8G}$, the angular defect vanishes and the geometry is smooth everywhere. The case $J = 0$ describes global AdS spacetime. Below this mass, the angular defect becomes an excess. When $0 > M > -\frac{|J|}{l}$, the whole spacetime contains closed time-like curves [5].

In the flat case, we have the same metric (59) without the $\frac{r^2}{l^2}$ term, or, in null coordinates, the right hand side of (47) with $\Theta = 8GM$ and $\Xi = 4GJ$, so that the charges are the same as in the AdS case, and thus again given by (58).

Identifying in both cases the solutions with the same charges then implies that AdS_3 corresponds to the Minkowski space-time $M = -\frac{1}{8G}$, $J = 0$, while $M = 0 = J$, the massless BTZ black hole, corresponds to the null orbifold (40). For $8GM = -\alpha^2 < 0$, we can supplement the change of variable that leads to (59) by the redefinition of the radial coordinate $\bar{r}^2 = \frac{r^2}{\alpha^2} + \frac{16G^2 J^2}{\alpha^4}$, bringing the line element into

$$ds^2 = - \left(\alpha dt - \frac{4GJ}{\alpha} d\varphi \right)^2 + d\bar{r}^2 + \alpha^2 \bar{r}^2 d\varphi^2, \quad (60)$$

which corresponds to a spinning particle in flat spacetime [48]. Under the redefinition $\bar{t} = \alpha t - \frac{4GJ}{\alpha} \varphi$ and $\bar{\phi} = \alpha \varphi$, the identification $(t, \varphi) \sim (t, \varphi + 2\pi)$ becomes $(\bar{t}, \bar{\phi}) \sim (\bar{t} - \frac{8\pi GJ}{\alpha}, \bar{\phi} + 2\pi\alpha)$. For $\alpha^2 < 1$, the spatial geometry corresponds to

a cone with deficit angle $\Delta\bar{\phi} = 2\pi(1 - |\alpha|)$, while for $\alpha^2 = 1$ and $J = 0$, the geometry corresponds to global Minkowski spacetime. For $\alpha^2 > 1$, the geometries possess an angular excess. This family may have arbitrarily negative values of M .

We now pass to study the case of positive M , and define $8GM = \alpha^2$. It is convenient to separate the analysis for $r < \frac{4G|J|}{|\alpha|}$ and $r > \frac{4G|J|}{|\alpha|}$. In the first region, we make the transformation $\bar{r}^2 = -\frac{r^2}{\alpha^2} + \frac{16G^2J^2}{\alpha^4}$ which produce the line element,

$$ds^2 = \left(-\alpha dt + \frac{4GJ}{\alpha} d\varphi\right)^2 + d\bar{r}^2 - \alpha^2 \bar{r}^2 d\varphi^2. \quad (61)$$

Therefore, inside this region, the direction ∂_φ is always time-like, generating closed time-like curves. It is a bounded time machine. For $r > \frac{4G|J|}{\alpha}$, the suitable transformation is $\bar{r}^2 = \frac{r^2}{\alpha^2} - \frac{16G^2J^2}{\alpha^4}$, and the metric turns out to be

$$ds^2 = -dT^2 + \left(\frac{4GJ}{\alpha}\right)^2 dX^2 + \alpha^2 T^2 d\varphi^2, \quad (62)$$

where we have defined $T = \bar{r}$, $X = \frac{\alpha^2}{4GJ}t + \varphi$ and we have made the identification $X \sim X + 2\pi$ and $\varphi \sim \varphi + 2\pi$. The outgoing spacetime is a cosmology whose spatial section is a 2-torus with radii $\frac{4GJ}{\alpha}$ and αT . Note that here the parameter M may not be identified with a mass. It is conjugate to a space-like translation generator, and corresponds to a momentum. When $J = 0$, the metric becomes $ds^2 = -dT^2 + dX^2 + \alpha^2 T^2 d\varphi^2$ with an unwrapped X -coordinate.

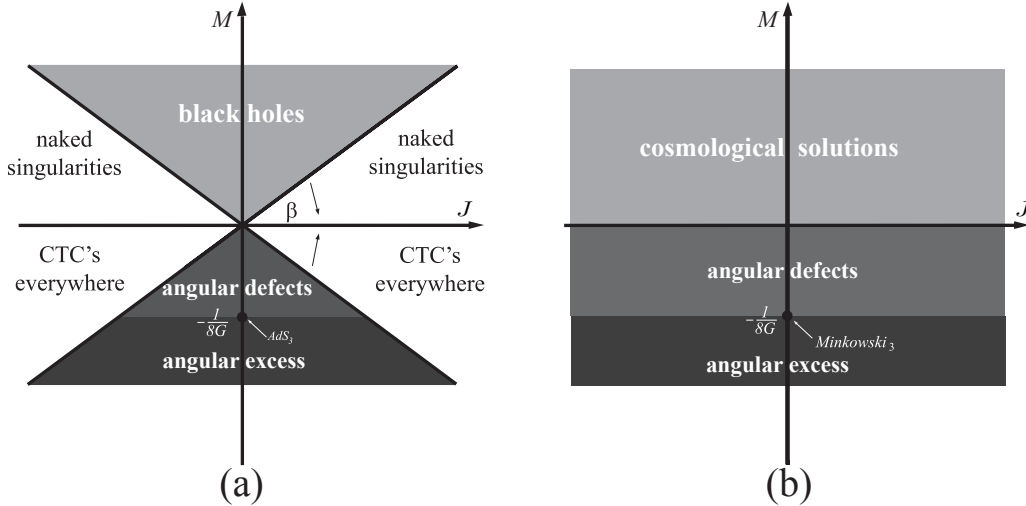


FIG. 2. Zero mode solutions of 2+1 gravity. Figure (a) depicts the case of non-vanishing cosmological constant. The slope β is given by $\tan \beta = \frac{1}{l}$. Figure (b) shows the limit $l \rightarrow \infty$, when β vanishes. No solutions are lost in the limit, but the horizon of the BTZ black holes gets pushed to infinity, hence the time coordinate becomes spatial everywhere and the line element describes the non-static, cosmological solution (62).

ACKNOWLEDGEMENTS

The work of G.B. is supported in part by the Fund for Scientific Research-FNRS (Belgium), by the Belgian Federal Science Policy Office through the Interuniversity Attraction Pole P6/11, by IISN-Belgium, by “Communauté française de Belgique - Actions de Recherche Concertées” and by Fondecyt Projects No. 1085322 and No. 1090753. The work of AG was partially supported by Fondecyt (Chile) Grant #1090753.

[1] A. Maloney and E. Witten, JHEP **1002**, 029 (2010), arXiv:0712.0155 [hep-th].

- [2] E. Witten, (2007), arXiv:0706.3359 [hep-th].
- [3] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998), hep-th/9711200.
- [4] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, Phys. Rept. **323**, 183 (2000), hep-th/9905111.
- [5] J. D. Brown and M. Henneaux, Commun. Math. Phys. **104**, 207 (1986).
- [6] A. Strominger, JHEP **02**, 009 (1998), arXiv:hep-th/9712251.
- [7] E. J. Martinec, (1998), arXiv:hep-th/9809021.
- [8] S. Carlip, Class. Quant. Grav. **15**, 3609 (1998), hep-th/9806026.
- [9] M. Bañados, in *Trends in Theoretical Physics II*, American Institute of Physics Conference Series, Vol. 484, edited by H. Falomir, R. E. Gamboa Saravi, and F. A. Schaposnik (1999) pp. 147–169.
- [10] K. Skenderis and S. N. Solodukhin, Phys. Lett. **B472**, 316 (2000), arXiv:hep-th/9910023.
- [11] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992), hep-th/9204099.
- [12] M. Rooman and P. Spindel, Class.Quant.Grav. **18**, 2117 (2001), arXiv:gr-qc/0011005 [gr-qc].
- [13] A. Achúcarro and P. K. Townsend, Phys. Lett. **B180**, 89 (1986).
- [14] E. Witten, Nucl. Phys. **B311**, 46 (1988).
- [15] E. Witten, Commun.Math.Phys. **121**, 351 (1989).
- [16] G. W. Moore and N. Seiberg, Phys. Lett. **B220**, 422 (1989).
- [17] S. Elitzur, G. W. Moore, A. Schwimmer, and N. Seiberg, Nucl. Phys. **B326**, 108 (1989).
- [18] O. Coussaert, M. Henneaux, and P. van Driel, Class. Quant. Grav. **12**, 2961 (1995), gr-qc/9506019.
- [19] M. Bañados, Phys. Rev. D **52**, 5816 (1995).
- [20] A. Balachandran, G. Bimonte, K. Gupta, and A. Stern, Int.J.Mod.Phys. **A7**, 4655 (1992), arXiv:hep-th/9110072 [hep-th].
- [21] E. Witten, “Talk given at Strings ’98,” available at <http://online.kitp.ucsb.edu/online/strings98/witten/>.
- [22] L. Susskind, (1998), arXiv:hep-th/9901079.
- [23] J. Polchinski, (1999), arXiv:hep-th/9901076.
- [24] J. de Boer and S. N. Solodukhin, Nucl. Phys. **B665**, 545 (2003), arXiv:hep-th/0303006.
- [25] S. N. Solodukhin, (2004), arXiv:hep-th/0405252.
- [26] A. Ashtekar, J. Bicak, and B. G. Schmidt, Phys. Rev. **D55**, 669 (1997), gr-qc/9608042.
- [27] G. Barnich and G. Compère, Class. Quant. Grav. **24**, F15 (2007), corrigendum: ibid 24 (2007) 3139, gr-qc/0610130.
- [28] H. Bondi, M. G. van der Burg, and A. W. Metzner, Proc. Roy. Soc. Lond. A **269**, 21 (1962).
- [29] R. K. Sachs, Proc. Roy. Soc. Lond. A **270**, 103 (1962).
- [30] G. Barnich and C. Troessaert, JHEP **05**, 062 (2010), arXiv:1001.1541 [hep-th].
- [31] P. Salomonson, B. Skagerstam, and A. Stern, Nucl.Phys. **B347**, 769 (1990).
- [32] A. Bagchi, R. Gopakumar, I. Mandal, and A. Miwa, JHEP **08**, 004 (2010), arXiv:0912.1090 [hep-th].
- [33] A. Bagchi, Phys. Rev. Lett. **105**, 171601 (2010).
- [34] A. Bagchi and R. Fareghbal, (2012), arXiv:1203.5795 [hep-th].
- [35] T. Regge and C. Teitelboim, Ann. Phys. **88**, 286 (1974).
- [36] J. D. Brown and M. Henneaux, J. Math. Phys. **27**, 489 (1986).
- [37] G. Barnich and F. Brandt, Nucl. Phys. **B633**, 3 (2002), hep-th/0111246.
- [38] G. Barnich and C. Troessaert, Proceedings of Science CNCFG **010** (2010), arXiv:1102.4632 [gr-qc].
- [39] R. Geroch, Communications in Mathematical Physics **13**, 180 (1969).
- [40] I. Robinson and A. Trautman, Proc. Roy. Soc. **A265**, 463 (1962).
- [41] R. Penrose, in *Differential Geometry and Relativity: A Volume in Honour of André Lichnerowicz on His 60th Birthday*, edited by M. Cahen and M. Flato (D. Reidel Publishing Compagny, 1976).
- [42] M. Blau, J. Figueroa-O’Farrill, C. Hull, and G. Papadopoulos, Classical and Quantum Gravity **19**, L87 (2002).
- [43] M. Blau, J. M. Figueroa-O’Farrill, and G. Papadopoulos, Class.Quant.Grav. **19**, 4753 (2002), arXiv:hep-th/0202111 [hep-th].
- [44] G. T. Horowitz and A. R. Steif, Phys.Lett. **B258**, 91 (1991).
- [45] M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. **D48**, 1506 (1993), gr-qc/9302012.
- [46] S. Deser and R. Jackiw, Annals Phys. **153**, 405 (1984).
- [47] O. Miskovic and J. Zanelli, Phys.Rev. **D79**, 105011 (2009), arXiv:0904.0475 [hep-th].
- [48] S. Deser, R. Jackiw, and G. ’t Hooft, Ann. Phys. **152**, 220 (1984).